J. Fluid Mech. (1968), *vol.* 32, *part* 1, *pp.* 21–28
Printed in Great Britain

Low Reynolds number heat transfer from a circular cylinder

By C. A. HIEBER AND B. GEBHART
Department of Thermal Engineering, Cornell University

(Received 4 April 1967 and in revised form 15 November 1967)

Theoretical results are obtained for forced heat convection from a circular cylinder at low Reynolds numbers. Consideration is given to the cases of a moderate and a large Prandtl number, the analysis in each case being based upon the method of matched asymptotic expansions. Comparison between the moderate Prandtl number theory and known experimental results indicates excellent agreement; no relevant experimental work has been found for comparison with the large Prandtl number theory.

Introduction

In the area of hot-wire anemometry, one encounters the phenomenon of forced convection from a heated circular cylinder at extremely low Reynolds numbers (R). Several theoretical analyses have treated this problem; however, none has made full use of the known velocity field, as $R\downarrow 0$, which was determined for this geometry by Kaplun (1957) and by Proudman & Pearson (1957).

In the present paper, use is made of the above velocity solution in order to solve the energy equation. The cases of a moderate and large Prandtl number (σ) are analysed. (The large σ analysis is subject to the constraint $\sigma R \ll 1$; such conditions can be expected to obtain in the application of hot-wire anemometry to low velocity flows such as natural convection base flow in liquids.) Natural convection and viscous dissipation are neglected and it is assumed that the fluid is of constant density and uniform transport properties. The method of matched asymptotic expansions is employed.

Cole & Roshko (1954) first considered this problem, applying Oseen's approximation to the energy equation (i.e. the velocity field was approximated by a uniform stream) and obtaining a solution for the temperature field in terms of an infinite series of modified Bessel and trigonometric functions, the coefficients being determined from the isothermal surface boundary condition. Based on this solution, Cole & Roshko computed the total heat transfer by considering the forced convection through a cylindrical surface (concentric to the body) of arbitrarily large radius. Expanding the resulting expression for small Peclet numbers (P, the product of R and σ), they obtained

$$N \sim \frac{2}{\ln(8/\gamma P)} \quad (P\downarrow 0) \tag{1}$$

(where N is the Nusselt number, the diameter, D, is the characteristic length and $\ln\gamma$ is Euler's constant $= 0{\cdot}577\ldots$). Illingworth (1963) employed the same technique as Cole & Roshko and obtained the next order term in this expansion:

$$N \sim \frac{2}{\ln(8/\gamma P)} - (\tfrac{1}{8}P)^2\left[8 + \frac{2}{\ln(8/\gamma P)^2}\right] \quad (P\downarrow 0). \tag{2}$$

Wood (1968) also treated this problem by employing Oseen's approximation; however, he first applied this approximation to the momentum equations, obtaining higher-order terms in the velocity field. His analysis of the thermal field therefore accounts for the thermal convection caused by these latter velocity components; following a lengthy and complicated analysis, he obtained the result:

$$N \sim \frac{2I_0(\frac{1}{4}P)}{K_0(\frac{1}{4}P)}\left\{1 - \frac{\lambda(\sigma)I_0(\frac{1}{4}P)}{K_0(\frac{1}{4}P)\,[I_0(\frac{1}{4}R)K_0(\frac{1}{4}R)+I_1(\frac{1}{4}R)K_1(\frac{1}{4}R)]} - \frac{\mu(\sigma)I_0(\frac{1}{4}P)}{K_0(\frac{1}{4}P)\,[I_0(\frac{1}{4}R)K_0(\frac{1}{4}R)+I_1(\frac{1}{4}R)K_1(\frac{1}{4}R)]^2}\right\} \quad (R\downarrow 0), \tag{3}$$

where I_n, K_n are modified Bessel functions of the second kind and $\lambda(\sigma)$, $\mu(\sigma)$ are constants (for a given σ) which must be obtained numerically. In particular, Wood found that $\lambda(0{\cdot}72) = 1{\cdot}38$, $\mu(0{\cdot}72) = 0{\cdot}40$.

The present paper treats the circular cylinder via the method of matched asymptotic expansions and considers the two limiting cases: (i) σ fixed, $R\downarrow 0$; (ii) $\sigma = (\frac{1}{2}R)^{-\alpha}$ $(0 < \alpha < 1)$, $R\downarrow 0$. For case (i), the leading term of the current expansion is identical to the result of Cole & Roshko, whereas the second term agrees closely with the second term in Wood's expansion. To the authors' knowledge, there is no existing theory for the large Prandtl number case treated in (ii).

Analysis

(i) $R\downarrow 0$, σ *fixed*

The temperature field is assumed to have the following asymptotic expansions:

$$T(r^*, \theta; R; \sigma) \sim \sum \delta_n(R) T_n(r, \theta; \sigma) \quad R\downarrow 0, \quad r \text{ fixed}, \tag{4}$$

$$T(r^*, \theta; R; \sigma) \sim \sum \Delta_n(R) \mathscr{T}_n(\rho, \theta; \sigma) \quad R\downarrow 0, \quad \rho \text{ fixed}, \tag{5}$$

where $T \equiv (t-t_\infty)/(t_w - t_\infty)$ (t being the temperature, subscripts w and ∞ indicating conditions at the wall and at infinity, respectively), r^* is the radial co-ordinate in physical space, θ is the angular co-ordinate measured from the downstream direction and r and ρ are the radial co-ordinates non-dimensionalized respectively with respect to the diameter of the body, D, and the viscous length, ν/U_∞ (ν being the kinematic viscosity and U_∞ the speed of the uniform stream). The 'Stokes' expansion, (4), is valid near the body and the 'Oseen' expansion, (5), is applicable at large distances from the body. As a result, the surface condition $T(\frac{1}{2}D, \theta; R; \sigma) = 1$ is imposed upon (4) and the uniform temperature condition at infinity is placed upon (5). Additional conditions on each of the expansions result from matching considerations [cf. Van Dyke (1964) or Kaplun & Lagerstrom (1957)].

For the circular cylinder, substitution of the Stokes and Oseen variables into the energy equation results in, respectively,

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = \sigma R\left(v_r\frac{\partial T}{\partial r}+v_\theta\frac{\partial T}{r\partial \theta}\right), \tag{6}$$

$$\frac{\partial^2 T}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial T}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 T}{\partial \theta^2} = \sigma\left(v_\rho\frac{\partial T}{\partial \rho}+v_\theta\frac{\partial T}{\rho\partial \theta}\right), \tag{7}$$

where v_r and v_θ are the velocity components, normalized by U_∞. Kaplun (1957) and Proudman & Pearson (1957) determined these velocity components in the limit $R\downarrow 0$. In the Stokes region, the velocity is $O(1/\ln R)$ whereas, in the Oseen region, the flow is that of a uniform stream perturbed by terms of $O(1/\ln R)$.

Based upon this velocity field, one sees that the leading term of the Oseen temperature expansion arises from a balance of thermal diffusion and convection by the uniform stream, the appropriate solution being

$$\mathscr{T}_0 = \exp\left(\tfrac{1}{2}\sigma\rho\cos\theta\right)K_0\left(\tfrac{1}{2}\sigma\rho\right). \tag{8}$$

Since (8) has a logarithmic singularity for small ρ (the matching region) and, due to the wall condition, the Stokes temperature expansion must be finite but non-zero, it follows that

$$\Delta_0(R) = O(1/\ln R) \quad (R\downarrow 0).$$

As a result, the forced convection of $\mathscr{T}_0$ by the Oseen velocity components of $O(1/\ln R)$ results in an effect of $O(1/\ln^2 R)$, implying that $\Delta_1(R)$ is of the same order. By induction,

$$\Delta_n(R) = O(1/\ln R)^{n+1} \quad (R\downarrow 0).$$

In the Stokes region, it is seen from (6) that thermal diffusion predominates. Making use of the isothermal wall condition, it follows that (4) must be of the form

$$T \sim 1+f(R)\ln 2r, \quad r \text{ fixed} \quad (R\downarrow 0), \tag{9}$$

where, from matching considerations,

$$f(R) = O(1/\ln R) \quad (R\downarrow 0).$$

The leading convective effect in the Stokes region is therefore of $O(R\ln^{-2}R)$, implying expansion (9) is applicable to the same order. Interestingly, although (9) is based upon pure conduction in the Stokes region, it is still dependent upon the thermal convection, as is evidenced by $f(R)$; this arises from matching considerations and indicates that the temperature field in the Stokes region is 'induced' by the velocity field in the Oseen region. In particular, for the case of no forced convection ($R = 0$), (9) reduces to a uniform temperature, indicating the well-known result that there does not exist a non-trivial solution for steady-state pure conduction from a heated circular cylinder in an unbounded medium.

Based upon the above considerations, a detailed analysis results in the following expansions:

$$T \sim \Delta \mathscr{T}_0 + \Delta^2 \mathscr{T}_1 + \Delta^3[-a_3(\sigma)\mathscr{T}_0 + F_3(\rho,\theta)] + O(1/\ln^4 R) \quad (\sigma \text{ and } \rho \text{ fixed, } R\downarrow 0), \quad (10)$$

$$T \sim 1 - \left(\Delta - \sum_3^\infty a_n(\sigma)\Delta^n\right)\ln 2r \quad (\sigma \text{ and } r \text{ fixed, } R\downarrow 0), \quad (11)$$

where $\Delta = (\ln 8/\gamma\sigma R)^{-1}$, $\mathscr{T}_0$ is as given in (8), $\mathscr{T}_1 \underset{\rho\downarrow 0}{\sim} a_3(\sigma) + O(\rho\ln^2\rho)$, $F_3 \underset{\rho\downarrow 0}{\sim} O(1)$, and $a_n(\sigma)$ is a constant (for a given σ) which must be determined numerically (cf. appendix). Only $a_3(\sigma)$ has been determined, the following values having been computed

$$a_3(0{\cdot}72) \approx 1{\cdot}38, \quad a_3(1{\cdot}0) \approx 1{\cdot}63, \quad a_3(6{\cdot}82) \approx 3{\cdot}42.$$

It should be indicated that the structure of expansions (10) and (11) is the same as the corresponding velocity expansions obtained by Kaplun (1957). Based upon (11), the heat transfer is

$$N \sim \frac{2}{\ln 8/\gamma\sigma R}\left[1 - \frac{a_3(\sigma)}{(\ln 8/\gamma\sigma R)^2}\right] + O[(\ln R)^{-4}] \quad (\sigma \text{ fixed, } R\downarrow 0). \quad (12)$$

(ii) $\sigma = (\tfrac{1}{2}R)^{-\alpha}$ $(0 < \alpha < 1)$, $R\downarrow 0$

With $\sigma = (\tfrac{1}{2}R)^{-\alpha}$, it is seen from the energy equation that the thermal convection and diffusion are comparable, as $R\downarrow 0$, if $(r^*/D)R^{1-\alpha}$ is held fixed. Therefore, $\hat{\rho} \equiv 2^\alpha R^{1-\alpha} r = 2^\alpha R^{-\alpha}\rho$ forms the 'Oseen' variable for the thermal field. Since, in the limit $R\downarrow 0$, $\hat{\rho}$ fixed, the momentum equation reduces to the Stokes equation, it follows that the velocity in the Oseen thermal region is based upon the Stokes stream function. This fact simplifies the analysis, enabling one to obtain a closed-form asymptotic solution.

Making the substitution $r = 2^{-\alpha}R^{\alpha-1}\hat{\rho}$ into the Stokes stream function, one finds that, in the Oseen thermal region, the leading term of the velocity field is a uniform stream of magnitude $U_\infty(1-\alpha)$. Hence, the 'effective' Reynolds number is $(1-\alpha)R$. [In the Oseen velocity region, thermal convection now predominates; since the leading term of the velocity field is a uniform stream, the temperature in this region is, to the order of the present analysis, uniform (i.e. T_∞).]

Using the same expansions for the temperature as in (4) and (5) [with $\hat{\rho}$, $\hat{\mathscr{T}}_n$ replacing ρ, $\mathscr{T}_n$ in (5)] and basing the velocity in both thermal regions upon the Stokes stream function, the resulting analysis parallels that of case (i). The results are:

$$T \sim \phi\hat{\mathscr{T}}_0 + \phi^2\hat{\mathscr{T}}_1 + \phi^3[-b_3(\alpha)\hat{\mathscr{T}}_0 + G_3(\hat{\rho},\theta)] + O(1/\ln^4 R) \quad (\hat{\rho} \text{ fixed, } \sigma = (\tfrac{1}{2}R)^{-\alpha}, R\downarrow 0), \quad (13)$$

$$T \sim 1 - \left(\phi - \sum_3^\infty b_n(\alpha)\phi^n\right)\ln 2r \quad (r \text{ fixed, } \sigma = (\tfrac{1}{2}R)^{-\alpha}, R\downarrow 0), \quad (14)$$

$$N \sim \frac{2}{\ln\left[\dfrac{8}{\gamma(1-\alpha)\sigma R}\right]}\left\{1 - \frac{b_3(\alpha)}{\ln^2\left[\dfrac{8}{\gamma(1-\alpha)\sigma R}\right]}\right\} + O[(\ln R)^{-4}] \quad (\sigma = (\tfrac{1}{2}R)^{-\alpha}, R\downarrow 0), \quad (15)$$

where $\hat{\mathscr{T}}_0 = \exp(\tfrac{1}{2}b\rho\cos\theta)K_0(\tfrac{1}{2}b\rho), \quad b = (1-\alpha),$

$$\phi = \left[\ln\frac{8}{(1-\alpha)\gamma\sigma R}\right]^{-1},$$

$$b_3(\alpha) = 2[\{(1-2\alpha)\ln 2 + (\tfrac{1}{2}-\ln\gamma)(1-\alpha)+\ln(1-\alpha)\}\beta_1+\beta_2],$$

$$\beta_1 \equiv \int_0^\infty xK_0^2(x)\,dx = \tfrac{1}{2}, \quad \beta_2 \equiv -\int_0^\infty x\ln xK_0^2(x)\,dx = \tfrac{1}{2}(1+\ln\tfrac{1}{2}\gamma),$$

$$\hat{\mathscr{T}}_1 \underset{\rho\downarrow 0}{\sim} b_3(\alpha) + O(\rho\ln^2\rho),$$

$$G_3(\rho,\theta) \underset{\rho\downarrow 0}{\sim} O(1).$$

Discussion

Comparison of the present result for the moderate σ case, (12), with that of Cole & Roshko, (1), indicates that the first term in the former is identical to the latter result. However, the second term in (12) does not agree with the result of Illingworth, (2). This disagreement is due to the fact that Illingworth did not consider the effect of higher-order components of the velocity field upon the temperature distribution (i.e. he tacitly assumed the Oseen approximation to be valid to all orders of R). In particular, the second term in the current result, (12), follows directly from considering the convection of $\mathscr{T}_0$ by the velocity components of $O[(\ln R)^{-1}]$ in the Oseen region.

The result of Wood, (3), is in an unnecessarily complicated form. By expanding the Bessel functions about $R = 0$, (3) reduces to

$$N \sim \frac{2}{\ln(8/\gamma P)}\left\{1-\frac{\lambda(\sigma)}{(\ln 8/\gamma P)(\ln 8/\gamma R)}-\frac{\mu(\sigma)-\frac{1}{2}\lambda(\sigma)}{(\ln 8/\gamma P)(\ln 8/\gamma R)^2}\right\}+O\left[\left(\frac{1}{\ln R}\right)^5\right], \quad (16)$$

as $R\downarrow 0$, the analysis of Wood being valid to $O[(\ln R)^{-4}]$. As expected, the leading term in (16) is identical to the result of Cole & Roshko. Comparison of the present moderate Prandtl number result, (12), with the first two terms in (16), indicates very close agreement, the only difference being that the second term in (16) contains the factor $(\ln 8/\gamma R)^{-1}$ rather than $(\ln 8/\gamma P)^{-1}$. This difference is actually of $O(1/\ln^4 R)$ and therefore does not constitute a disagreement between the theories, (12) being valid to only $O(1/\ln^3 R)$. It is noteworthy that, by using the method of matched asymptotic expansions in the present work, the labour required in obtaining the higher-order term was greatly reduced. Specifically, in the present analysis it was only necessary to obtain the explicit behaviour of $\hat{\mathscr{T}}_1$ as $\rho\downarrow 0$ (i.e. in the matching region). However, the very fact that $\hat{\mathscr{T}}_1$ was not determined explicitly throughout the Oseen region meant that, in the present paper, the temperature expansion could not be extended beyond this term. This shortcoming is of little practical significance since, based upon (16), the term of $O(1/\ln^4 R)$ contributes less than 3 % of the total heat transfer for $R < 0{\cdot}40$.

A comparison of the above theories (for $\sigma = 0{\cdot}72$) with the empirical correlation obtained by Collis & Williams (1959) is shown in figure 1. The experimental results were obtained from heated wires in air under conditions for which the natural convection was negligible. It is seen from the graph that the present

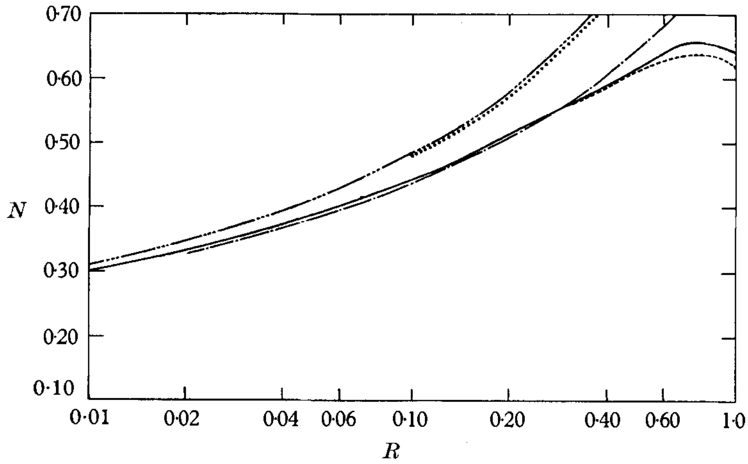

FIGURE 1. Forced heat convection from a circular cylinder in air (σ = 0·72). ——, present theory, equation (12); —···—, Cole & Roshko, equation (1); ....., Illingworth, equation (2); —·—, Collis & Williams, experimental correlation; - - - -, Wood, equation (16).

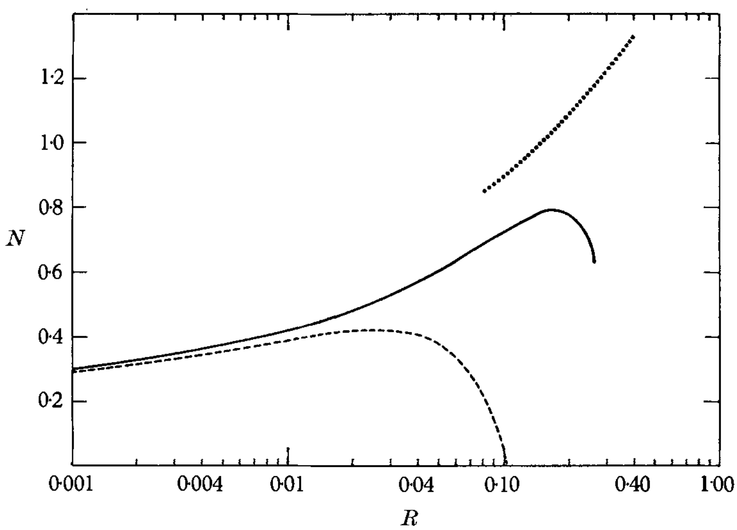

FIGURE 2. Forced heat convection from a circular cylinder in water (σ = 6·82). ——, present high σ theory, equation (15); - - -, present moderate σ theory, equation (12); ····, Piret *et al.* experimental correlation.

result, (12), is almost coincident with that of Wood, (16); in the range $R < 0{\cdot}40$, these two theories agree to within 1 %. Comparison with the experimental curve indicates that the theories of Wood and the present paper are in closest agreement with the correlation of Collis & Williams. In the range $R = 0{\cdot}02$ (lower limit of experiment) to $R = 0{\cdot}40$, the deviation between the current result, (12), and the experimental correlation is less than 3 %.

In figure 2, the high Prandtl number result, (15), is shown for the case $\sigma = 6{\cdot}82$ (corresponding to water). For comparison, the moderate Prandtl number theory, (12), is also plotted. The only relevant data that has been found for this case is that obtained by Piret, James & Stacy (1947); their experiment involved the use of 10^{-3} in. diameter wires in water. However, their results are limited to $R > 0{\cdot}08$ and correspond to temperature differences $(t_w - t_\infty)$ of 50–130 °F, making comparison with a uniform σ theory suspect. Nevertheless, it is evident from figure 2 that the applicability of the current large σ theory is less restricted (in R) than the moderate σ result, the former appearing to be valid up to about $R = 0{\cdot}04$. Complete verification of the theory is evidently precluded by the absence of data in this low R régime.

The authors acknowledge the support of the National Science Foundation through grants GP127 and GK1963.

Appendix. Determination of $a_3(\sigma)$

Letting $\mathscr{T}_1 \equiv \exp(\frac{1}{2}\sigma\rho\cos\theta)F(\rho,\theta)$, the equation for $\mathscr{T}_1$ becomes

$$(\nabla^2_\rho - \tfrac{1}{4}\sigma^2)F = f(\rho,\theta;\sigma),$$

where

$$f(\rho,\theta;\sigma) = (\sigma^2/\rho)[-K_1(\tfrac{1}{2}\sigma\rho) + \cos\theta K_0(\tfrac{1}{2}\sigma\rho)] + \tfrac{1}{2}\sigma^2\exp(\tfrac{1}{2}\rho\cos\theta)[K_1(\tfrac{1}{2}\rho) \times K_1(\tfrac{1}{2}\sigma\rho) - \cos\theta K_1(\tfrac{1}{2}\rho)K_0(\tfrac{1}{2}\sigma\rho) + \cos\theta K_0(\tfrac{1}{2}\rho)K_1(\tfrac{1}{2}\sigma\rho) - K_0(\tfrac{1}{2}\rho)K_0(\tfrac{1}{2}\sigma\rho)].$$

The appropriate Green's function is

$$G(\bar{\rho},\bar{\rho}') = \left(-\frac{1}{2\pi}\right)K_0(\tfrac{1}{2}\sigma|\bar{\rho}-\bar{\rho}'|),$$

where $\bar{\rho}$ signifies the (vector) displacement of a given point (ρ,θ) from the origin.

$$\begin{aligned} F(\rho=0) &= \int_0^\infty\int_0^{2\pi} G(0,\rho')f(\rho',\theta')\rho'\,d\theta'\,d\rho' \\ &= \frac{\sigma^2}{2\pi}\int_0^\infty\int_0^{2\pi}\bigg[\frac{1}{\rho'}K_1(\tfrac{1}{2}\sigma\rho')K_0(\tfrac{1}{2}\sigma\rho') - \frac{\cos\theta}{\rho'}K_0^2(\tfrac{1}{2}\sigma\rho') \\ &\quad + \tfrac{1}{2}\exp(\tfrac{1}{2}\rho'\cos\theta)\{-K_1(\tfrac{1}{2}\rho')K_1(\tfrac{1}{2}\sigma\rho')K_0(\tfrac{1}{2}\sigma\rho') + \cos\theta' K_1(\tfrac{1}{2}\rho')K_0^2(\tfrac{1}{2}\sigma\rho') \\ &\quad - \cos\theta' K_0(\tfrac{1}{2}\rho')K_1(\tfrac{1}{2}\sigma\rho')K_0(\tfrac{1}{2}\sigma\rho') + K_0(\tfrac{1}{2}\rho')K_0^2(\tfrac{1}{2}\sigma\rho')\}\bigg]\rho'\,d\theta'\,d\rho'. \end{aligned}$$

That is,

$$\begin{aligned} F(\rho=0) &= \frac{\sigma^2}{2}\int_0^\infty [2K_1(\tfrac{1}{2}\sigma\rho')K_0(\tfrac{1}{2}\sigma\rho') - \rho' I_0(\tfrac{1}{2}\rho')K_1(\tfrac{1}{2}\rho')K_1(\tfrac{1}{2}\sigma\rho')K_0(\tfrac{1}{2}\sigma\rho') \\ &\quad + \rho' I_1(\tfrac{1}{2}\rho')K_1(\tfrac{1}{2}\rho')K_0^2(\tfrac{1}{2}\sigma\rho') - \rho' I_1(\tfrac{1}{2}\rho')K_0(\tfrac{1}{2}\rho')K_1(\tfrac{1}{2}\sigma\rho')K_0(\tfrac{1}{2}\sigma\rho') \\ &\quad + \rho' I_0(\tfrac{1}{2}\rho')K_0(\tfrac{1}{2}\rho')K_0^2(\tfrac{1}{2}\sigma\rho')]\,d\rho'. \end{aligned} \tag{17}$$

This latter integral must be evaluated numerically. It should be noted that this procedure for determining $a_3(\sigma)$ is completely analogous to that employed by Kaplun (1957) in determining the corresponding velocity field. Comparing (17) with the corresponding integral obtained by Wood (1968), it is found that the two are identical since three of the terms in (17) cancel:

$$2 - xI_0(\tfrac{1}{2}x)K_1(\tfrac{1}{2}x) - xI_1(\tfrac{1}{2}x)K_0(\tfrac{1}{2}x) = 0.$$

REFERENCES

COLE, J. & ROSHKO, A. 1954 Heat transfer from wires at Reynolds numbers in the Oseen range. *Proceedings of Heat Transfer and Fluid Mechanics Institute*, 13–23.

COLLIS, D. C. & WILLIAMS, M. J. 1959 Two dimensional convection from heated wires at low Reynolds numbers. *J. Fluid Mech.* **6**, 357–84.

ILLINGWORTH, C. R. 1963 Flow at small Reynolds number; in *Laminar Boundary Layers* (ed. L. Rosenhead). Oxford: Clarendon Press.

KAPLUN, S. 1957 Low Reynolds number flow past a circular cylinder. *J. Math. Mech.* **6**, 595–603.

KAPLUN, S. & LAGERSTROM, P. A. 1957 Asymptotic expansions of Navier–Stokes solutions for small Reynolds numbers. *J. Math. Mech.* **6**, 585–93.

PIRET, E. L., JAMES, W. & STACY, M. 1947 Heat transfer from fine wires to water. *Indust. and Eng. Chem.* **39**, 1098–1103.

PROUDMAN, J. & PEARSON, J. R. A. 1957 Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. *J. Fluid Mech.* **2**, 237–62.

VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics*. New York: Academic Press.

WOOD, W. W. 1968 Calculations for anemometry with fine hot wires. *J. Fluid Mech.* **32**, 9.